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Third-Order Neutral Differential Equations with Non-Canonical Forms: Novel Oscillation Theorems

Barakah Almarri ¹ , Belal Batiha ^{2,*} , Omar Bazighifan ^{3,4,*}  and Fahd Masood ⁵ 

¹ General Studies Department, Jubail Industrial College, 8244 Rd Number 6, Al Huwaylat, Al Jubail 35718, Saudi Arabia; marribj@rcjy.edu.sa

² Mathematics Department, Faculty of Science and Information Technology, Jadara University, Irbid 21110, Jordan

³ Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen

⁴ Jadara Research Center, Jadara University, Irbid 21110, Jordan

⁵ Department of Mathematics, Faculty of Education and Science, University of Saba Region, Marib 14400, Yemen; fahdmasoud22@gmail.com

* Correspondence: b.bateha@jadara.edu.jo (B.B.); o.bazighifan@gmail.com (O.B.)

Abstract: This paper explores the asymptotic and oscillatory properties of a class of third-order neutral differential equations with multiple delays in a non-canonical form. The main objective is to simplify the non-canonical form by converting it to a canonical form, which reduces the complexity of the possible cases of positive solutions and their derivatives from four cases in the non-canonical form to only two cases in the canonical form, which facilitates the process of inference and development of results. New criteria are provided that exclude the existence of positive solutions or Kneser-type solutions for this class of equations. New criteria that guarantee the oscillatory behavior of all solutions that satisfy the conditions imposed on the studied equation are also derived. This work makes a qualitative contribution to the development of previous studies in the field of neutral differential equations, as it provides new insights into the oscillatory behavior of neutral equations with multiple delays. To confirm the strength and effectiveness of the results, three examples are included that highlight the accuracy of the derived criteria and their practical applicability, which enhances the value of this research and expands the scope of its use in the field.

Keywords: oscillatory behavior; non-oscillatory behavior; neutral differential equations; third-order; non-canonical case; multiple delays

MSC: 34C10; 34K11



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1. Introduction

In this study, we focus on the oscillatory properties of the third-order linear neutral differential equation represented as

$$\left(r_2(s) \left(r_1(s) z'(s) \right)' \right)' + \sum_{i=1}^n q_i(s) x(\eta_i(s)) = 0, \quad (1)$$

where $z(s) := x(s) + p(s)x(\tau(s))$, $s \geq s_0$. We consider the following fundamental assumptions throughout the paper:

(H₁) $\tau, \eta_i \in C^1([s_0, \infty), \mathbb{R})$, $\eta_i(s) \leq s$, $\eta_i'(s) > 0$, $\left(\eta_i^{-1}(s) \right)' \geq \eta_0 > 0$, $\tau'(s) \geq \tau_0 > 0$, and $\lim_{s \rightarrow \infty} \tau(s) = \lim_{s \rightarrow \infty} \eta_i(s) = \infty$, $i = 1, 2, \dots, n$.

(H₂) $p, q \in C([s_0, \infty), (0, \infty))$, $0 \leq p(s) \leq p_0 < \infty$ and $q(s)$ does not vanish identically.

(H₃) $r_1 \in C^2([s_0, \infty), (0, \infty))$, $r_2 \in C^1([s_0, \infty), (0, \infty))$, and (1) is in non-canonical case, that is

$$\int_{s_0}^{\infty} \frac{1}{r_1(\ell)} d\ell < \infty \text{ and } \int_{s_0}^{\infty} \frac{1}{r_2(\ell)} d\ell < \infty. \quad (2)$$

$$(H_4) \quad \tau(s) \leq s \text{ and } \tau \circ \eta_i = \eta_i \circ \tau, i = 1, 2, \dots, n.$$

A function $x \in C^3([S_x, \infty), \mathbb{R})$, $S_x \geq s_0$, is said to be a solution of (1) which has the property $r_2(r_1 z')' \in C^1[S_x, \infty)$, and it satisfies Equation (1) for all $x \in [S_x, \infty)$. We consider only those solutions x of (1) which exist on some half-line $[S_x, \infty)$ and satisfy the condition

$$\sup\{|x(s)| : s \geq S\} > 0, \text{ for all } S \geq S_x.$$

For the sake of simplicity, we define the operators:

$$L_0 z = z, L_1 z = r_1 z', L_2 z = r_2(r_1 z')', \text{ and } L_3 z = (r_2(r_1 z')')'.$$

A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be non-oscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Functional differential equations (FDEs) are equations in which the current value of variables depends on past or future states, making them essential for studying systems influenced by their historical behavior. Key types of these equations include delay differential equations, neutral functional differential equations, integro-differential equations, and advanced differential equations. These equations are vital for modeling complex systems in fields such as physics, biology, and engineering. For example, in ecology, FDEs help analyze population dynamics based on past states, while in control theory, they manage feedback systems to ensure stability. As modern systems grow increasingly complex, ongoing research in FDEs is crucial for developing theoretical foundations and numerical methods to solve real-world problems (see [1–6]).

Neutral differential equations (NDEs) refer to a specific subset of functional differential equations (FDEs) where derivatives are influenced by both the current values of the function and its derivatives at earlier times. This unique characteristic distinguishes NDEs from traditional FDEs and establishes a distinct analytical framework. The relationship between NDEs and FDEs is pivotal, as they often arise in systems where both past values and rates of change affect future states. The significance of NDEs is particularly evident in fields such as control theory and signal processing, where they describe systems with memory effects. For instance, in mechanical systems with inertia, acceleration may depend on both the current position and velocity. This relationship highlights the importance of NDEs in accurately modeling and simulating dynamic systems. Furthermore, the study of NDEs complements that of FDEs, as understanding one often provides valuable insights into the other. Thus, the exploration of neutral differential equations enriches the theoretical framework and enhances practical applications across various scientific and engineering disciplines (see [7–14]).

Oscillatory theorems play a critical role in the analysis of functional and neutral differential equations, as they establish essential criteria for determining whether solutions exhibit oscillatory behavior. Understanding this behavior is vital for assessing the stability and long-term dynamics of various systems, especially in engineering contexts where predicting oscillation versus steady-state behavior influences design and operational decisions. These theorems often utilize advanced mathematical techniques, such as comparison principles and integral inequalities, to derive significant results. Moreover, the relationship between oscillatory behavior and the inherent characteristics of these differential equations underscores a rich research area, particularly regarding how delays and memory effects can shape the oscillatory nature of solutions. This intersection is key to advancing knowledge in both theoretical and applied mathematics (see [15–21]).

Recent studies have explored advanced numerical methods, particularly radial basis function (RBF) networks and neural networks, for solving neutral delay differential equa-

tions (NDDEs), demonstrating significant improvements in both precision and stability. For instance, Saeed [22] highlighted the effectiveness of RBF networks as a powerful strategy for addressing the inherent complexities of NDDEs, offering enhanced numerical efficiency in dealing with delayed systems. Building on this, Noorizadegan et al. [23] refined the RBF approach by employing the fictitious point method, resulting in notable advancements in precision and stability through optimal selection of the shape parameter, a critical factor in solving complex delay differential equations. Additionally, Vinodhbhai and Dubey [24] introduced a novel application of orthogonal neural networks for solving NDDEs, achieving remarkable accuracy in capturing the intricate dynamics of delayed behaviors.

While even-order neutral differential equations have received considerable attention, the study of odd-order NDEs has also gained traction, reflecting a growing recognition of their importance. Interested readers are encouraged to consult several key studies in this area, including the pioneering works by Parhi and Das [25] and Parhi and Padhi [26,27]. Subsequent contributions include studies by Baculikova and Džurina [28] and Džurina [29], which focus on oscillation criteria and asymptotic behavior. Noteworthy recent advancements can be found in the research conducted by Bohner et al. [30] and Chatzarakis et al. [31,32], which provide significant insights into oscillation conditions for third-order delay differential equations. The works of Moaaz et al. [33] further explore Kneser-type solutions and oscillatory behavior in various contexts. Finally, recent contributions by Ma-sood et al. [34,35] present more effective criteria for assessing the asymptotic and oscillatory behavior of solutions.

In recent years, there has been considerable progress in the study of the oscillatory behavior of higher-order differential equations, particularly third-order differential equations with delays. Bohner et al. [36] extended Hanan's Kneser-type oscillation criterion, initially developed for ordinary differential equations of the form

$$x'''(s) + q(s)x(s) = 0,$$

to DDEs of the form

$$x'''(s) + q(s)x(\eta(s)) = 0. \quad (3)$$

This extension was shown to retain sharpness when applied to the delay Euler DEs

$$x'''(s) + \frac{q_0}{s^3}x(\lambda s) = 0, \lambda \in (0, 1), s \geq 1,$$

and in [37] the same authors extend the results and complete the study about (3).

Chatzarakis et al. [38] introduced new criteria to assess the oscillatory behavior of third-order NDEs of the form

$$z'''(\ell) + q(\ell)x^\alpha(\eta(\ell)) = 0.$$

They developed rigorous conditions demonstrating the nonexistence of Kneser-type solutions in these equations, pushing the boundaries of oscillation theory for NDEs.

Through the use of Riccati transformation techniques, Saker [39] established conditions that ensure every solution to the NDEs of the form

$$\left(r_2(s)(r_1(s)z'(s))' \right)' + q(s)f(x(s-\eta)) = 0, \quad (4)$$

is oscillatory. This study contributed valuable insights into the oscillation of solutions in nonlinear DEs.

In the non-canonical case, Grace et al. [40] developed new criteria for oscillation in third-order delay differential equations, specifically equations of the form

$$\left(r_2(s)(r_1(s)x'(s))' \right)' + q(s)x(\eta(s)) = 0. \quad (5)$$

These criteria were extended by Baculikova [41], who employed an appropriate substitution to transform (5) into the canonical form. This transformation enabled the introduction of new oscillation criteria, offering a more comprehensive understanding of oscillation behavior in non-canonical equations.

Further advancements were made by Nithyakala et al. [42] and Purushothaman et al. [43], who derived asymptotic and oscillatory results for third-order NDEs of the form

$$\left(r_2(s)(r_1(s)z'(s))'\right)' + q(s)x(\eta(s)) = 0,$$

with non-canonical operators and they used the same transform that in [41].

This research aims to build upon these foundational studies by deriving new oscillation criteria for a specific class of neutral third-order differential equations with multiple delays. By extending the results of previous studies, particularly those of [42,43], we transform the investigated equations from non-canonical form into canonical form. The results obtained in this study are consistent with previous findings, offering refined criteria for oscillatory solutions when $n = 1$. The comparison method with first-order equations was employed to achieve these results, marking a significant contribution to the field of differential equation oscillation theory.

The rest of this paper is organized as follows: Section 2 presents the essential definitions and foundational lemmas required to simplify the mathematical operations in the subsequent sections. In Section 3, we establish the conditions that guarantee the nonexistence of N-Kneser solutions in the case of (N_0) . Section 4 outlines the specific conditions that ensure the nonexistence of positive solutions in the case of (N_2) . In Section 5, we combine the results from the previous sections to formulate comprehensive oscillatory criteria for the equation under consideration. Section 6 provides examples that validate and illustrate the effectiveness of our results. Finally, Section 7 provides a conclusion that highlights the key contributions of this work and suggests avenues for future research.

2. Preliminary Results

This section introduces definitions, lemmas, and assumptions essential for simplifying the mathematical calculations used throughout this paper. These elements lay the groundwork for understanding the more complex results presented in later sections. For the sake of brevity and clarity, we define the key terms and notations as follows:

$$\begin{aligned}\pi_1(s) &:= \int_s^\infty \frac{1}{r_1(\ell)} d\ell, \quad \pi_2(s) := \int_s^\infty \frac{1}{r_2(\ell)} d\ell, \quad \pi_3(s) := \int_s^\infty \frac{\pi_2(\ell)}{r_1(\ell)} d\ell, \quad \pi_*(s) := \int_s^\infty \frac{\pi_1(\ell)}{r_2(\ell)} d\ell \\ \kappa_1(s) &:= \frac{r_1(s)\pi_3^2(s)}{\pi_*(s)}, \quad \text{and} \quad \kappa_2(s) := \frac{r_2(s)\pi_*(s)}{\pi_3(s)}. \\ \mu_1(s) &:= \int_{s_0}^s \frac{1}{\kappa_1(\ell)} d\ell, \quad \mu_2(s) := \int_{s_0}^s \frac{1}{\kappa_2(\ell)} d\ell, \quad \mu_3(s) := \int_{s_0}^s \frac{\mu_2(\ell)}{\kappa_1(\ell)} d\ell, \\ \mu_1(\varsigma, \varrho) &:= \int_\varrho^\varsigma \frac{1}{\kappa_1(\ell)} d\ell, \quad \mu_2(\varsigma, \varrho) := \int_\varrho^\varsigma \frac{1}{\kappa_2(\ell)} d\ell, \quad \mu_3(\varsigma, \varrho) := \int_\varrho^\varsigma \frac{\mu_2(\ell, \varrho)}{\kappa_1(\ell)} d\ell \\ \tilde{g}_i(s) &:= \min\{g_i(s), g_i(\tau(s))\}, \quad \hat{g}_i(s) := \min\left\{g_i\left(\eta_i^{-1}(s)\right), g_i\left(\eta_i^{-1}(\tau(s))\right)\right\}, \\ G_1(s) &:= \sum_{i=1}^n \tilde{g}_i(s)\pi_3(\eta_i(s)), \quad G_2(s) := \pi_3(s) \sum_{i=1}^n \hat{g}_i(s), \\ G_3(s) &:= \sum_{i=1}^n g_i(s)(1 - p(\eta_i(s)))\pi_3(\eta_i(s)),\end{aligned}$$

and

$$\eta_{\max}(s) := \max\{\eta_i(s), i = 1, 2, \dots, n\}, \quad \eta_{\min}(s) := \min\{\eta_i(s), i = 1, 2, \dots, n\}$$

Lemma 1 ([44]). Let γ be a ratio of two odd positive integers. $\kappa > 0$ and B are constants. Then

$$Bu - \kappa u^{(\gamma+1)/\gamma} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{\kappa^\gamma}, \quad \kappa > 0. \quad (6)$$

Lemma 2 ([45]). Let x be an eventually positive solution of Equation (1). Then there exists $s_1 \geq s_0$ such that the associated function z can be classified into one of the following cases:

$$\begin{aligned} \text{Case (C}_0\text{)} &: z > 0, L_1 z < 0, L_2 z > 0, L_3 z < 0, \\ \text{Case (C}_1\text{)} &: z > 0, L_1 z < 0, L_2 z < 0, L_3 z < 0, \\ \text{Case (C}_2\text{)} &: z > 0, L_1 z > 0, L_2 z > 0, L_3 z < 0, \\ \text{Case (C}_3\text{)} &: z > 0, L_1 z > 0, L_2 z < 0, L_3 z < 0, \end{aligned}$$

for $s \geq s_1$.

To establish conditions for the oscillation of Equation (1), we must eliminate all four of these cases. Moreover, for the nonexistence of Kneser-type solutions, Cases (C₀) and (C₁) must be ruled out. However, when Equation (1) is transformed into its canonical form, the number of classes for non-oscillatory solutions is reduced from four to two, significantly simplifying the analysis.

We begin by transforming the original equation into an equivalent canonical form, as established in [41]. The transformed equation is expressed as follows:

$$\left(\kappa_2(s) \left(\kappa_1(s) \left(\frac{z(s)}{\pi_3(s)} \right)' \right)' \right)' + \pi_*(s) \sum_{i=1}^n q_i(s) x(\eta_i(s)) = 0. \quad (7)$$

By introducing the substitution $w(s) = \frac{z(s)}{\pi_3(s)}$, and defining $g_i(s) := \pi_*(s)q_i(s)$, we can simplify the equation to

$$\left(\kappa_2(s) (\kappa_1(s) w'(s))' \right)' + \sum_{i=1}^n g_i(s) x(\eta_i(s)) = 0, \quad (8)$$

with

$$\int_{s_0}^{\infty} \frac{1}{\kappa_1(\ell)} d\ell < \infty \text{ and } \int_{s_0}^{\infty} \frac{1}{\kappa_2(\ell)} d\ell < \infty.$$

This transformation indicates that the solution of the noncanonical equation is equivalent to the solution of the canonical form.

Next, we define

$$D_0 w = w, D_1 w = r_1 w', D_2 w = r_2 \left((r_1 w')' \right)', \text{ and } D_3 w = \left(w_2 \left((r_1 w')' \right) \right)'.$$

We can now conclude that the noncanonical neutral differential Equation (1) has an eventually positive solution if and only if the canonical Equation (8) exhibits the same behavior. This simplifies the examination to two distinct classes of eventually positive solutions.

Lemma 3 ([46]). Assume that x is an eventually positive solution of Equation (8). Then there exists $s_1 \geq s_0$ such that w is one of the following cases:

$$\begin{aligned} \text{Case (N}_0\text{)} &: w > 0, \kappa_1 w' < 0, \kappa_2 (\kappa_1 w')' > 0, \left(\kappa_2 (\kappa_1 w')' \right)' < 0, \\ \text{Case (N}_2\text{)} &: w > 0, \kappa_1 w' > 0, \kappa_2 (\kappa_1 w')' > 0, \left(\kappa_2 (\kappa_1 w')' \right)' < 0. \end{aligned}$$

We denote the sets Ω_0 and Ω_2 as the collections of eventually, positive solutions satisfying the conditions of Case (N_0) and Case (N_2) , respectively.

Definition 1 ([47]). The solutions x whose corresponding function $w \in N_0$ are called Kneser-type solutions.

Definition 2 ([47]). Equation (1) has property A if and only if any nonoscillatory solution x is Kneser-type and $\lim_{s \rightarrow \infty} x(s) = 0$.

3. Conditions for the Absence of N-Kneser Solutions

In this section, we outline particular conditions that ensure the nonexistence of N-Kneser solutions fulfilling the case (N_0) within Category Ω_0 .

Theorem 1. Let $\zeta(s) \in C([s_0, \infty), (0, \infty))$ such that $\eta_{\max}(s) < \zeta(s)$ and $\tau^{-1}(\zeta(s)) < s$. If the differential equation

$$\omega'(s) + \frac{\tau_0}{\tau_0 + p_0} G_1(s) \mu_3(\zeta(s), \eta_{\max}(s)) \omega(\tau^{-1}(\zeta(s))) = 0, \quad (9)$$

is oscillatory, then it follows that $\Omega_0 = \emptyset$.

Proof. Let $x \in \Omega_0$ such that $x(s) > 0$ and $x(\eta_i(s)) > 0$ for $s \geq s_1 \geq s_0$. This leads to

$$(-1)^k w^{(k)}(s) > 0, \text{ for } k = 0, 1, 2, 3. \quad (10)$$

From Equation (8), it follows that

$$\begin{aligned} 0 &\geq \frac{p_0}{\tau(t)} D_3 w(\tau(s)) + \pi_*(s) \sum_{i=1}^n p_0 g_i(\tau(s)) x(\eta_i(\tau(s))) \\ &\geq \frac{p_0}{\tau_0} D_3 w(\tau(s)) + \pi_*(s) \sum_{i=1}^n p_0 g_i(\tau(s)) x(\eta_i(\tau(s))). \end{aligned} \quad (11)$$

Combining (8) and (11), we obtain

$$\begin{aligned} 0 &\geq D_3 w(s) + \frac{p_0}{\tau_0} D_3 w(\tau(s)) + \sum_{i=1}^n [g_i(s) x(\eta_i(s)) + p_0 g_i(\tau(s)) x(\eta_i(\tau(s)))] \\ &\geq D_3 w(s) + \frac{p_0}{\tau_0} D_3 w(\tau(s)) + \sum_{i=1}^n \tilde{g}_i(s) [x(\eta_i(s)) + p_0 x(\eta_i(\tau(s)))]. \end{aligned} \quad (12)$$

Based on the definition of z , we can express it as

$$z(\eta_i(s)) = x(\eta_i(s)) + p(\eta_i(s)) x(\tau(\eta_i(s))) \leq x(\eta_i(s)) + p_0 x(\tau(\eta_i(s))).$$

By using the latter inequality in (12), we obtain

$$0 \geq D_3 w(s) + \frac{p_0}{\tau_0} D_3 w(\tau(s)) + \sum_{i=1}^n \tilde{g}_i(s) z(\eta_i(s)). \quad (13)$$

Since $w(s) = z(s) / \pi_3(s)$, then

$$0 \geq D_3 w(s) + \frac{p_0}{\tau_0} D_3 w(\tau(s)) + \sum_{i=1}^n \tilde{g}_i(s) \pi_3(\eta_i(s)) w(\eta_i(s)).$$

Since $w'(s) < 0$, then

$$0 \geq D_3 w(s) + \frac{p_0}{r_0} D_3 w(\tau(s)) + w(\eta_{\max}(s)) \sum_{i=1}^n \tilde{g}_i(s) \pi_3(\eta_i(s)).$$

That is

$$\left(D_2 w(s) + \frac{p_0}{r_0} D_2 w(\tau(s)) \right)' + G_1(s) w(\eta_{\max}(s)) \leq 0. \quad (14)$$

On the other hand, it follows from the monotonicity of $D_2 w$ that

$$\begin{aligned} -D_1 w(\varrho) &\geq D_1 w(\varsigma) - D_1 w(\varrho) = \int_{\varrho}^{\varsigma} \frac{D_2 w(\ell)}{\kappa_2(\ell)} d\ell \geq D_2 w(\varsigma) \int_{\varrho}^{\varsigma} \frac{1}{\kappa_2(\ell)} d\ell \\ &= D_2 w(\varsigma) \mu_2(\varsigma, \varrho). \end{aligned} \quad (15)$$

Integrating (15) from ϱ to ς , we have

$$\begin{aligned} w(\varrho) &\geq -w(\varsigma) + w(\varrho) = \int_{\varrho}^{\varsigma} \frac{D_2 w(\ell) \mu_2(\varsigma, \varrho)}{\kappa_1(\ell)} d\ell \geq D_2 w(\varsigma) \int_{\varrho}^{\varsigma} \frac{\mu_2(\varsigma, \varrho)}{\kappa_1(\ell)} d\ell \\ &= D_2 w(\varsigma) \mu_3(\varsigma, \varrho). \end{aligned} \quad (16)$$

Then

$$w(\varrho) \geq D_2 w(\varsigma) \mu_3(\varsigma, \varrho). \quad (17)$$

Thus, we have

$$w(\eta_{\max}(s)) \geq D_2 w(\zeta(s)) \mu_3(\zeta(s), \eta_{\max}(s))$$

which, by virtue of (14) yields

$$0 \geq \left(D_2 w(s) + \frac{p_0}{r_0} D_2 w(\tau(s)) \right)' + G_1(s) \mu_3(\zeta(s), \eta_{\max}(s)) D_2 w(\zeta(s)). \quad (18)$$

Now, set

$$\omega(s) = D_2 w(s) + \frac{p_0}{r_0} D_2 w(\tau(s)) > 0.$$

Since $D_2 w$ is a non-increasing function, it follows that

$$\omega(s) \leq D_2 w(\tau(s)) \left(1 + \frac{p_0}{r_0} \right),$$

which can be rewritten as

$$D_2 w(\zeta(s)) \geq \frac{r_0}{r_0 + p_0} \omega(\tau^{-1}(\zeta(s))). \quad (19)$$

Substituting (19) into (18), we obtain that ω satisfies the following differential inequality

$$\omega'(s) + \frac{r_0}{r_0 + p_0} G_1(s) \mu_3(\zeta(s), \eta_{\max}(s)) \omega(\tau^{-1}(\zeta(s))) \leq 0.$$

According to ([48], Theorem 1), this implies that (9) also has a positive solution, leading to a contradiction. Therefore, the proof is concluded. $\square$

Corollary 1. Let $\zeta(s) \in C([s_0, \infty), (0, \infty))$ such that $\eta_{\max}(s) < \zeta(s)$ and $\tau^{-1}(\zeta(s)) < s$. If

$$\liminf_{s \rightarrow \infty} \int_{\tau^{-1}(\zeta(s))}^s G_1(\ell) \mu_3(\zeta(\ell), \eta_{\max}(\ell)) d\ell > \frac{r_0 + p_0}{r_0 e}, \quad (20)$$

then $\Omega_0 = \emptyset$.

Theorem 2. Let $\delta(s) \in C([s_0, \infty), (0, \infty))$ such that $\delta(s) < s$ and $\eta_{\max}(s) < \tau(\delta(s))$. If

$$\limsup_{s \rightarrow \infty} \mu_3(\tau(\delta(s)), \eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell > \frac{\tau_0 + p_0}{\tau_0}, \quad (21)$$

then $\Omega_0 = \emptyset$.

Proof. Following the same method used in the proof of Theorem 1, we derive the inequality

$$\left(D_2 w(s) + \frac{p_0}{\tau_0} D_2 w(\tau(s)) \right)' + G_1(s) w(\eta_{\max}(s)) \leq 0.$$

By integrating this inequality over the interval $(\eta(s), s)$ and applying the fact that w is a decreasing function, we obtain

$$\begin{aligned} D_2 w(\delta(s)) + \frac{p_0}{\tau_0} D_2 w(\tau(\delta(s))) &\geq D_2 w(s) + \frac{p_0}{\tau_0} D_2 w(\tau(s)) + \int_{\delta(s)}^s G_1(\ell) w(\eta_{\max}(\ell)) d\ell \\ &\geq w(\eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell. \end{aligned}$$

Since $\tau(\delta(s)) < \tau(s)$ and $D_3 w \leq 0$, we have

$$D_2 w(\tau(\delta(s))) \left(1 + \frac{p_0}{\tau_0} \right) \geq w(\eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell. \quad (22)$$

By utilizing (17) with $\varsigma = \tau(\delta(s))$ and $\varrho = \eta_{\max}(s)$ into (22), we deduce that

$$D_2 w(\tau(\delta(s))) \left(1 + \frac{p_0}{\tau_0} \right) \geq D_2 w(\tau(\delta(s))) \mu_3(\tau(\delta(s)), \eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell.$$

That is

$$\frac{\tau_0 + p_0}{\tau_0} \geq \mu_3(\tau(\delta(s)), \eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell.$$

Taking the lim sup of both sides of the inequality reveals a contradiction with (21). Thus, we can conclude the proof. $\square$

Corollary 2. Letting $\delta(s) = \tau(s)$ in Theorem 2. If $\eta(s) < \tau(\tau(s))$, such that

$$\limsup_{s \rightarrow \infty} \mu_3(\tau(\tau(s)), \eta_{\max}(s)) \int_{\tau(s)}^s G_1(\ell) d\ell > \frac{\tau_0 + p_0}{\tau_0}, \quad (23)$$

holds, then $\Omega_0 = \emptyset$.

Theorem 3. Assume that $\eta_{\max}(\tau(s)) < s$ holds. If the differential equation

$$\Psi'(s) + \frac{\eta_0 \tau_0}{\tau_0 + p_0} \mu_3(\tau(s), s) G_2(s) \pi(s) \Psi(\eta_{\max}(s)) = 0, \quad (24)$$

is oscillatory, then $\Omega_0 = \emptyset$.

Proof. Let $x \in \Omega_0$ such that $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\eta_i(s)) > 0$ for $s \geq s_1 \geq s_0$. This leads to

$$(-1)^k z^{(k)}(s) > 0, \text{ for } k = 0, 1, 2, 3.$$

From Equation (8), it follows that

$$\begin{aligned} 0 &\geq \frac{1}{(\eta_i^{-1}(s))'} D_3 w(\eta_i^{-1}(s)) + \sum_{i=1}^n g_i(\eta_i^{-1}(s)) x(s) \\ &\geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \sum_{i=1}^n g_i(\eta_i^{-1}(s)) x(s) \end{aligned}$$

Similarly,

$$\begin{aligned} 0 &\geq \frac{p_0}{(\eta_i^{-1}(\tau(s)))'} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n p_0 g_i(\eta_i^{-1}(\tau(s))) x(\tau(s)) \\ &\geq \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n p_0 g_i(\eta_i^{-1}(\tau(s))) x(\tau(s)). \end{aligned}$$

Combining the above inequalities yields that

$$\begin{aligned} 0 &\geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \sum_{i=1}^n g_i(\eta_i^{-1}(s)) x(s) \\ &\quad + \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n p_0 g_i(\eta_i^{-1}(\tau(s))) x(\tau(s)) \\ &\geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n [g_i(\eta_i^{-1}(s)) x(s) + p_0 g_i(\eta_i^{-1}(\tau(s))) x(\tau(s))] \\ &\geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n \hat{g}_i(s) [(x(s) + p_0 x(\tau(s)))]. \end{aligned}$$

From the definition of z , we have

$$0 \geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + \sum_{i=1}^n \hat{g}_i(s) z(s).$$

Since $w(s) = z(s) / \pi(s)$, then

$$0 \geq \frac{1}{\eta_0} D_3 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_3 w(\eta_i^{-1}(\tau(s))) + w(s) \pi_3(s) \sum_{i=1}^n \hat{g}_i(s).$$

That is,

$$0 \geq \left(\frac{1}{\eta_0} D_2 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_2 w(\eta_i^{-1}(\tau(s))) \right)' + G_2(s) w(s). \quad (25)$$

Now, we set

$$\Psi(s) = \frac{1}{\eta_0} D_2 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_2 w(\eta_i^{-1}(\tau(s))). \quad (26)$$

From the assumption (H_4) and the observation that $D_2 w$ is non-increasing, it follows that

$$\Psi(s) \leq \frac{D_2 w(\eta_i^{-1}(\tau(s)))}{\eta_0} \left(1 + \frac{p_0}{\tau_0} \right) \leq \frac{D_2 w(\eta_{\max}^{-1}(\tau(s)))}{\eta_0} \left(1 + \frac{p_0}{\tau_0} \right). \quad (27)$$

By using (17) with $\varsigma = \tau(s)$ and $\varrho = s$ and (27), we can deduce that

$$w(s) \geq D_2 w(\tau(s)) \mu_3(\tau(s), s) \geq \Psi(\eta_{\max}(s)) \mu_3(\tau(s), s) \left(\frac{\eta_0 \tau_0}{\tau_0 + p_0} \right).$$

From the definition of Ψ and applying the above inequality in (25), we obtain

$$0 \geq \Psi'(s) + \frac{\eta_0 \tau_0}{\tau_0 + p_0} \mu_3(\tau(s), s) G_2(s) \pi(s) \Psi(\eta_{\max}(s)).$$

According to ([48], Theorem 1), this implies that (24) also has a positive solution, leading to a contradiction. Therefore, the proof is concluded. $\square$

Corollary 3. Suppose that $\eta_i(\tau(s)) < s$, $i = 1, 2, \dots, n$ holds. If

$$\liminf_{s \rightarrow \infty} \int_{\eta_{\max}(s)}^s \mu_3(\tau(\ell), \ell) G_2(\ell) d\ell > \frac{\tau_0 + p_0}{\eta_0 \tau_0 e}, \quad (28)$$

then $\Omega_0 = \emptyset$.

4. Absence of Solutions in Class N_2

In this section, we focus on the asymptotic and monotonic characteristics of the positive solutions for the equation under investigation. Furthermore, we outline specific conditions that ensure the nonexistence of positive solutions that meet the case (N_2) within Category Ω_2 .

Lemma 4. Suppose that $x \in \Omega_2$. Then, eventually,

$$x(s) > (1 - p(s))z(s),$$

and Equation (8) eventually becomes

$$D_3 w(s) + G_3(s) w(\eta(s)) \leq 0. \quad (29)$$

Proof. Since

$$z(s) = x(s) + p(s)x(\tau(s)),$$

then $z(s) \geq x(s)$ and

$$x(s) = z(s) - p(s)x(\tau(s)) \geq z(s) - p(s)z(\tau(s)).$$

Since $z(s)$ is increasing, then

$$x(s) \geq (1 - p(s))z(s). \quad (30)$$

From (8), we have

$$D_3 w(s) = - \sum_{i=1}^n g_i(s) x(\eta_i(s)) \leq - \sum_{i=1}^n g_i(s) (1 - p(\eta_i(s))) z(\eta_i(s)).$$

Since $w = z/\pi$, then

$$D_3 w(s) \leq - \sum_{i=1}^n g_i(s) (1 - p(\eta_i(s))) \pi_3(\eta_i(s)) w(\eta_i(s)).$$

Since $w'(s) > 0$, then

$$D_3w(s) \leq -w(\eta_{\min}(s)) \sum_{i=1}^n g_i(s)(1 - p(\eta_i(s)))\pi_3(\eta_i(s)).$$

That is,

$$D_3w(s) \leq -G_3(s)w(\eta_{\min}(s)).$$

Therefore, the proof is concluded. $\square$

Lemma 5. Assume that $x \in \Omega_2$. Then

- (i) $D_1w(s) \geq \mu_2(s)D_2w(s)$ and $D_1w(s)/\mu_2(s)$ is decreasing;
- (ii) $w(s) \geq \mu_3(s)D_2w(s)$.

Proof. Let $z(s) \in \Omega_2$. Since $D_2w(s)$ is decreasing, then

$$D_1w(s) \geq \int_{s_0}^s D_2w(\ell) \frac{1}{\kappa_2(\ell)} d\ell \geq D_2w(s) \int_{s_0}^s \frac{1}{\kappa_2(\ell)} d\ell = \mu_2(s)D_2w(s). \quad (31)$$

Therefore,

$$\left(\frac{D_1w(s)}{\mu_2(s)} \right)' = \frac{\mu_2(s)D_2w(s) - D_1w(s)}{\kappa_2(s)\mu_2^2(s)} \leq 0.$$

Now,

$$w(s) = \int_{s_0}^s \frac{D_1w(\ell)}{\mu_2(\ell)} \frac{\mu_2(\ell)}{\kappa_1(\ell)} d\ell \geq \frac{D_1w(s)}{\mu_2(s)} \int_{s_0}^s \frac{\mu_2(\ell)}{\kappa_1(\ell)} d\ell = \mu_3(s) \frac{D_1w(s)}{\mu_2(s)},$$

and from (31), we find

$$w(s) \geq \mu_3(s)D_2w(s).$$

$\square$

Theorem 4. Assume that there is a $\rho \in C^1([s_0, \infty), (0, \infty))$ such that

$$\limsup_{s \rightarrow \infty} \int_{s_0}^s \left(\rho(\ell)G_3(\ell) - \frac{\kappa_1(\eta_{\min}(\ell))[\rho'(\ell)]^2}{4\rho(\ell)\eta'(\ell)\mu_2(\eta_{\min}(\ell))} \right) d\ell = \infty. \quad (32)$$

Then $\Omega_2 = \emptyset$.

Proof. Suppose the contrary, i.e., that $x \in \Omega_2$. We define

$$H(s) = \rho(s) \frac{D_2w(s)}{w(\eta_{\min}(s))}. \quad (33)$$

Then $w(s) > 0$. Differentiating (33), we have

$$\begin{aligned} H'(s) &= \rho'(s) \frac{D_2w(s)}{w(\eta_{\min}(s))} + \rho(s) \frac{D_3w(s)}{w(\eta_{\min}(s))} - \eta'_{\min}(s) \rho(s) \frac{D_2w(s)w'(\eta_{\min}(s))}{w^2(\eta_{\min}(s))} \\ &\leq \frac{\rho'(s)}{\rho(s)} H(s) - \rho(s)G_3(s) - \eta'_{\min}(s)H(s) \frac{w'(\eta_{\min}(s))}{w(\eta_{\min}(s))}. \end{aligned} \quad (34)$$

Using Lemma 5, we obtain

$$H'(s) \leq \frac{\rho'(s)}{\rho(s)} H(s) - \rho(s)G_3(s) - \eta'_{\min}(s)H(s) \frac{\mu_2(\eta_{\min}(s))D_2w(\eta_{\min}(s))}{\kappa_1(\eta_{\min}(s))w(\eta_{\min}(s))}.$$

Since D_2w is decreasing, then

$$\begin{aligned} H'(s) &\leq \frac{\rho'(s)}{\rho(s)}H(s) - \rho(s)G_3(s) - \frac{\eta'_{\min}(s)\mu_2(\eta_{\min}(s))}{\kappa_1(\eta_{\min}(s))}H(s)\frac{D_2w(s)}{w(\eta(s))} \\ &= \frac{\rho'(s)}{\rho(s)}H(s) - \rho(s)G_3(s) - \frac{\eta'(s)\mu_2(\eta_{\min}(s))}{\rho(s)\kappa_1(\eta_{\min}(s))}H^2(s) \end{aligned} \quad (35)$$

Using Lemma 1 where $B = \rho'(s)/\rho(s)$, $\kappa = \eta'(s)\mu_2(\eta_{\min}(s))/\rho(s)\kappa_1(\eta_{\min}(s))$, and $u = H$, we have

$$\frac{\rho'(s)}{\rho(s)}H(s) - \frac{\eta'(s)\mu_2(\eta_{\min}(s))}{\rho(s)\kappa_1(\eta_{\min}(s))}H^2(s) \leq \frac{\kappa_1(\eta_{\min}(s))[\rho'(s)]^2}{4\rho(s)\eta'(s)\mu_2(\eta_{\min}(s))}.$$

Substituting the previous inequality into (35), we obtain

$$H'(s) \leq -\rho(s)g_3(s) + \frac{\kappa_1(\eta_{\min}(s))[\rho'(s)]^2}{4\rho(s)\eta'(s)\mu_2(\eta_{\min}(s))}. \quad (36)$$

Integrating (36) from s_1 to s , we have

$$\int_{s_1}^s \left(\rho(\ell)G_3(\ell) - \frac{\kappa_1(\eta_{\min}(\ell))[\rho'(\ell)]^2}{4\rho(\ell)\eta'(\ell)\mu_2(\eta_{\min}(\ell))} \right) d\ell \leq H(s_1),$$

which contradicts (32). $\square$

Theorem 5. *If the differential equation*

$$\xi'(s) + \left(\frac{r_0 + p_0}{r_0} \right) G_1(s)\mu_3(\eta_{\min}(s))\xi(r^{-1}(\eta_{\min}(s))) = 0, \quad (37)$$

is oscillatory, then $\Omega_2 = \emptyset$.

Proof. Assume the contrary that $x \in \Omega_2$. Using (13), we see that

$$0 \geq D_3w(s) + \frac{p_0}{r_0}D_3w(r(s)) + \sum_{i=1}^n \tilde{q}_i(s)z(\eta_i(s)).$$

Since $w(s) = z(s)/\pi_3(s)$, and $w' > 0$, then

$$0 \geq D_3w(s) + \frac{p_0}{r_0}D_3w(r(s)) + w(\eta_{\min}(s)) \sum_{i=1}^n \tilde{q}_i(s)\pi_3(\eta_i(s)).$$

That is,

$$\left(D_2w(s) + \frac{p_0}{r_0}D_2w(r(s)) \right)' + G_1(s)w(\eta_{\min}(s)) \leq 0.$$

Using (ii) in Lemma 5, we find

$$\left(D_2w(s) + \frac{p_0}{r_0}D_2w(r(s)) \right)' + G_1(s)\mu_3(\eta_{\min}(s))D_2w(\eta_{\min}(s)) \leq 0.$$

Assuming the following:

$$\xi(s) = D_2w(s) + \frac{p_0}{r_0}D_2w(r(s)) \leq \left(1 + \frac{p_0}{r_0} \right) D_2w(r(s)). \quad (38)$$

Thus,

$$\zeta(\tau^{-1}(s)) \leq \left(1 + \frac{p_0}{\tau_0}\right) D_2 w(s),$$

and

$$\zeta(\tau^{-1}(\eta_{\min}(s))) \leq \left(1 + \frac{p_0}{\tau_0}\right) D_2 w(\eta_{\min}(s)). \quad (39)$$

Substituting from (39) into (38), we conclude that

$$\zeta'(s) + \left(\frac{\tau_0 + p_0}{\tau_0}\right) \mu_3(\eta_{\min}(s)) G_1(s) \zeta(\tau^{-1}(\eta_{\min}(s))) \leq 0.$$

According to ([48], Theorem 1), this implies that (37) also has a positive solution, leading to a contradiction. Therefore, the proof is concluded. $\square$

Corollary 4. *If*

$$\liminf_{s \rightarrow \infty} \int_{\tau^{-1}(\eta_{\min}(s))}^s G_1(\ell) \mu_3(\eta_{\min}(\ell)) d\ell > \frac{\tau_0 + p_0}{\tau_0 e}, \quad (40)$$

then $\Omega_2 = \emptyset$.

Theorem 6. *If the differential equation*

$$B'(s) + \left(\frac{\eta_0 \tau_0}{\tau_0 + p_0}\right) G_2(s) \pi(s) \mu(s) B(\eta(\tau^{-1}(s))) = 0, \quad (41)$$

is oscillatory, then $\Omega_2 = \emptyset$.

Proof. Assume the contrary that $x \in \Omega_2$. Using (25) and (ii) in Lemma 5, we see that

$$0 \geq \left(\frac{1}{\eta_0} D_2 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_2 w(\eta_i^{-1}(\tau(s)))\right)' + G_2(s) \mu_3(s) D_2 w(s). \quad (42)$$

Assume the following:

$$B(s) = \frac{1}{\eta_0} D_2 w(\eta_i^{-1}(s)) + \frac{p_0}{\eta_0 \tau_0} D_2 w(\eta_i^{-1}(\tau(s))) \quad (43)$$

It follows from $\tau(s) < s$ that

$$B(s) \leq \frac{\tau_0 + p_0}{\eta_0 \tau_0} D_2 w(\eta_{\min}^{-1}(\tau(s))).$$

Thus,

$$\frac{\eta_0 \tau_0}{\tau_0 + p_0} B(\eta_{\min}(\tau^{-1}(s))) \leq D_2 w(s). \quad (44)$$

Substituting (43) and (44) into (42), we conclude that

$$B'(s) + \left(\frac{\eta_0 \tau_0}{\tau_0 + p_0}\right) G_2(s) \mu_3(s) B(\eta_{\min}(\tau^{-1}(s))) \leq 0. \quad (45)$$

In view of ([48], Theorem 1), we have that (41) also has a positive solution, i.e., a contradiction. Thus, the proof is complete. $\square$

Corollary 5. *If*

$$\liminf_{s \rightarrow \infty} \int_{\eta_{\min}(\tau^{-1}(s))}^s G_2(\ell) \mu_3(\ell) d\ell > \frac{\tau_0 + p_0}{\eta_0 \tau_0 e}, \quad (46)$$

then $\Omega_2 = \emptyset$.

5. Oscillatory Criteria

In this section, we combine the results obtained in the previous sections to derive oscillatory criteria for the examined equation. By summarizing these findings, we establish a framework for understanding the conditions under which oscillatory behavior is guaranteed.

Theorem 7. Assume that each one of the following conditions is satisfied separately:

- (i) Conditions (20) and (40);
- (ii) Conditions (21) and (40);
- (iii) Conditions (28) and (46);
- (iv) Conditions (23) and (40);
- (v) Conditions (20) and (32);
- (vi) Conditions (23) and (46);
- (vii) Conditions (23) and (32).

Then, Equation (1) exhibits oscillatory behavior.

Proof. To demonstrate this, we begin by proving the first case (i), and the same method applies to all other cases.

Assume for the sake of contradiction that x is an eventually positive solution of Equation (8). According to Lemma 3, we can deduce that there are two potential scenarios regarding the behavior of w and its derivatives. By applying Corollaries 1 and 4, we observe that the conditions (20) and (40) imply the absence of solutions to Equation (8) that fulfill the requirements of cases (N_0) and N_2 , respectively. Consequently, this leads us to conclude that our initial assumption must be false, thereby confirming that the solutions of Equation (8) are indeed oscillatory.

The proofs for the remaining cases proceed in the same manner, with the corresponding conditions substituted accordingly. This ensures that oscillatory behavior is consistently maintained across all scenarios. Thus, the proof is complete. $\square$

6. Illustrative Examples

In this section, we include three examples to illustrate and confirm the validity of our results.

Example 1. Consider the third-order neutral differential equation

$$\left(s^\alpha \left(s^\beta (x(s) + p_0 x(\tau_0 s))' \right)' \right)' + \sum_{i=1}^n \frac{q_0}{s^{3-\alpha-\beta}} x(\eta_i s) = 0, \quad \alpha > 1, \beta > 1, \quad (47)$$

where $z(s) = x(s) + p_0 x(\tau_0 s)$, $\alpha > 0$, $\beta > 0$, $0 < p_0 < 1$. By comparing this equation with Equation (1), we can observe that

$$r_1(s) = s^\beta, \quad r_2(s) = s^\alpha, \quad \eta_i(s) = \eta_i s, \quad \tau(s) = \tau_0 s, \quad q_i(s) = \frac{q_0}{s^{3-\alpha-\beta}}.$$

As a result, we obtain

$$\pi_1(s) = \frac{s^{1-\beta}}{\beta-1}, \quad \pi_2(s) = \frac{s^{1-\alpha}}{\alpha-1}, \quad \pi_3(s) = \frac{s^{2-\alpha-\beta}}{(\alpha-1)(\alpha+\beta-2)}, \quad \pi_*(s) = \frac{s^{2-\alpha-\beta}}{(\beta-1)(\alpha+\beta-2)},$$

$$\kappa_1(s) = \frac{(\beta-1)}{(\alpha-1)^2(\alpha+\beta-2)} s^{2-\alpha}, \quad \kappa_2(s) = \frac{(\alpha-1)}{(\beta-1)^2(\alpha+\beta-2)} s^{2-\beta}.$$

Now, we can transform (47) into canonical form

$$\left(s^{2-\beta} \left(s^{2-\alpha} \left(\frac{z(s)}{\pi_3(s)} \right)' \right)' \right)' + (\alpha-1)(\alpha+\beta-2) \sum_{i=1}^n \frac{q_0}{s} x(\eta_i s) = 0,$$

which, setting $w(s) = z(s)/\pi_3(s)$, transforms to

$$\left(s^{2-\beta}\left(s^{2-\alpha}w'(s)\right)'\right)' + (\alpha-1)(\alpha+\beta-2)\frac{q_0}{s}\sum_{i=1}^n x(\eta_i s) = 0.$$

Moreover,

$$\mu_1(s) = \frac{s^{\alpha-1}}{\alpha-1}, \mu_2(s) = \frac{s^{\beta-1}}{\beta-1}, \mu_3(s) = \frac{s^{\beta+\alpha-2}}{(\beta-1)(\beta+\alpha-2)},$$

and

$$\eta_{\max}(s) = \eta_{\max}s = \max\{\eta_i s, i = 1, 2, \dots, n\}, \eta_{\min}(s) = \eta_{\min}s = \min\{\eta_i s, i = 1, 2, \dots, n\}.$$

Example 2. Consider the special case of (47) that $\alpha = \beta = 2$, which simplifies to

$$\left(s^2\left(s^2(x(s) + p_0 x(\tau_0 s))'(s)\right)'\right)' + \sum_{i=1}^n q_0 s x(\eta_i s) = 0, \quad (48)$$

where $q_0 > 0$. Comparing this equation with (1), we observe that

$$r_1(s) = r_2(s) = s^2, \eta_i(s) = \eta_i s, \tau(s) = \tau_0 s, q_i(s) = q_0 s.$$

Consequently, we obtain

$$\pi_1(s) = \pi_2(s) = \frac{1}{s}, \pi_3(s) = \pi_*(s) = \frac{1}{2s^2}.$$

By defining $w(s) = z(s)/\pi_3(s)$, we transform (48) into its canonical form:

$$w'''(s) + \frac{2q_0}{s}\sum_{i=1}^n x(\eta_i s) = 0.$$

Additionally, we have the following relations:

$$\mu_1(s) = \mu_2(s) = s, \mu_3(s) = \frac{s^2}{2},$$

$$g_i(s) = \frac{2q_0}{s}, \tilde{g}_i(s) = \frac{2q_0}{s}, \hat{g}_i(s) = \frac{2q_0\eta_i}{s}, G_1(s) = \frac{q_0}{s^3}\sum_{i=1}^n \frac{1}{\eta_i^2},$$

$$G_2(s) = \frac{q_0}{s^3}\sum_{i=1}^n \eta_i, G_3(s) = \frac{q_0}{s^3}(1-p_0)\sum_{i=1}^n \frac{1}{\eta_i^2}, i = 1, 2, \dots, n,$$

and

$$\eta_{\max}(s) = \eta_{\max}s = \max\{\eta_i s, i = 1, 2, \dots, n\}, \eta_{\min}(s) = \eta_{\min}s = \min\{\eta_i s, i = 1, 2, \dots, n\}.$$

From Condition (20), we obtain

$$\begin{aligned} \liminf_{s \rightarrow \infty} \int_{\tau^{-1}(\zeta(s))}^s G_1(\ell) \mu_3(\zeta(\ell), \eta_{\max}(\ell)) d\ell &= \liminf_{s \rightarrow \infty} \int_{\frac{\zeta_0}{\tau_0}s}^s \frac{q_0}{\ell^3} \sum_{i=1}^n \frac{1}{\eta_i^2} \frac{(\zeta_0 - \eta_{\max})^2}{2} \ell^2 d\ell \\ &= \frac{q_0(\zeta_0 - \eta_{\max})^2}{2} \sum_{i=1}^n \frac{1}{\eta_i^2} \ln \frac{\tau_0}{\zeta_0}, \end{aligned}$$

which holds if

$$q_0 > \frac{2}{(\zeta_0 - \eta_{\max})^2 \ln \frac{\tau_0}{\zeta_0} \sum_{i=1}^n \frac{1}{\eta_i^2}} \frac{(\tau_0 + p_0)}{\tau_0 e}. \quad (49)$$

Condition (21) leads to

$$\begin{aligned} \limsup_{s \rightarrow \infty} \mu_3(\tau(\delta(s)), \eta_{\max}(s)) \int_{\delta(s)}^s G_1(\ell) d\ell &= \limsup_{s \rightarrow \infty} \frac{(\tau_0 \delta_0 - \eta_{\max})^2 s^2}{2} \int_{\delta_0 s}^s \frac{q_0}{\ell^3} \sum_{i=1}^n \frac{1}{\eta_i^2} d\ell \\ &= \frac{(\tau_0 \delta_0 - \eta_{\max})^2 (1 - \delta_0) q_0}{4 \delta_0} \sum_{i=1}^n \frac{1}{\eta_i^2}, \end{aligned}$$

which holds if

$$q_0 > \frac{4 \delta_0 (\tau_0 + p_0)}{(\tau_0 \delta_0 - \eta_{\max})^2 (1 - \delta_0) \tau_0 \sum_{i=1}^n \frac{1}{\eta_i^2}}. \quad (50)$$

Similarly, Condition (23) is satisfied if

$$q_0 > \frac{4(\tau_0 + p_0)}{(\tau_0^2 - \eta_{\max})^2 (1 - \tau_0) \sum_{i=1}^n \frac{1}{\eta_i^2}}. \quad (51)$$

Condition (28) gives

$$\begin{aligned} \liminf_{s \rightarrow \infty} \int_{\eta_{\max}(s)}^s \mu_3(\tau(\ell), \ell) G_2(\ell) d\ell &= \liminf_{s \rightarrow \infty} \int_{\eta_{\max} s}^s \frac{(\tau_0 - 1)^2 \ell^2}{2} \frac{q_0}{\ell^3} \sum_{i=1}^n \eta_i d\ell \\ &= \frac{q_0 (\tau_0 - 1)^2 \sum_{i=1}^n \eta_i}{2} \ln \frac{1}{\eta_{\max}} > \frac{\tau_0 + p_0}{\eta_0 \tau_0 e} \end{aligned}$$

and is satisfied when

$$q_0 > \frac{2}{(\tau_0 - 1)^2 (\sum_{i=1}^n \eta_i) \ln \frac{1}{\eta_{\max}}} \frac{\tau_0 + p_0}{\eta_0 \tau_0 e}. \quad (52)$$

In Condition (32) where $\rho(s) = s^2$, we find

$$\begin{aligned} \limsup_{s \rightarrow \infty} \int_{s_0}^s \left(\rho(\ell) G_3(\ell) - \frac{\kappa_1(\eta_{\min}(\ell)) [\rho'(\ell)]^2}{4 \rho(\ell) \eta'_{\min}(\ell) \mu_2(\eta_{\min}(\ell))} \right) d\ell \\ = \limsup_{s \rightarrow \infty} \int_{s_0}^s \left(\ell^2 \frac{q_0}{\ell^3} (1 - p_0) \sum_{i=1}^n \frac{1}{\eta_i^2} - \frac{4 \ell^2}{4 \ell^2 \eta_{\min} \eta_{\min} \ell} \right) d\ell \\ = \limsup_{s \rightarrow \infty} \int_{s_0}^s \left(q_0 (1 - p_0) \sum_{i=1}^n \frac{1}{\eta_i^2} - \frac{1}{\eta_{\min}^2} \right) \frac{1}{\ell} d\ell = \infty, \end{aligned}$$

which holds if

$$q_0 > \frac{1}{\eta_{\min}^2 (1 - p_0) \sum_{i=1}^n \frac{1}{\eta_i^2}}. \quad (53)$$

Similarly, Condition (40) yields

$$\begin{aligned} \liminf_{s \rightarrow \infty} \int_{\tau^{-1}(\eta_{\min}(s))}^s G_1(\ell) \mu_3(\eta_{\min}(\ell)) d\ell &= \liminf_{s \rightarrow \infty} \int_{\frac{\eta_{\min}}{\tau_0} s}^s \frac{q_0}{\ell^3} \sum_{i=1}^n \frac{1}{\eta_i^2} \frac{\eta_{\min}^2 \ell^2}{2} d\ell \\ &= \frac{1}{2} q_0 \eta_{\min}^2 \left(\sum_{i=1}^n \frac{1}{\eta_i^2} \right) \ln \frac{\tau_0}{\eta_{\min}}, \end{aligned}$$

which is satisfied when

$$q_0 > \frac{2}{\eta_{\min}^2 \left(\sum_{i=1}^n \frac{1}{\eta_i^2} \right) \ln \frac{\tau_0}{\eta_{\min}}} \frac{(\tau_0 + p_0)}{\tau_0 e}. \quad (54)$$

Moreover, Condition (46) gives

$$\begin{aligned} \liminf_{s \rightarrow \infty} \int_{\eta_{\min}(\tau^{-1}(s))}^s G_2(\ell) \mu_3(\ell) d\ell &= \liminf_{s \rightarrow \infty} \int_{\frac{\eta_{\min}}{\tau_0} s}^s \frac{q_0}{\ell^3} \sum_{i=1}^n \eta_i \frac{\ell^2}{2} d\ell \\ &= \frac{q_0}{2} \sum_{i=1}^n \eta_i \ln \frac{\tau_0}{\eta_{\min}} > \frac{\tau_0 + p_0}{\eta_0 \tau_0 e}, \end{aligned}$$

which holds if

$$q_0 > \frac{2}{(\sum_{i=1}^n \eta_i) \ln \frac{\tau_0}{\eta_{\min}}} \frac{\tau_0 + p_0}{\eta_0 \tau_0 e}. \quad (55)$$

Now, according to Theorem (7), we establish the oscillatory behavior of (48) through specific pairs of conditions. Specifically, (49) pairs with (54), (50) pairs with (54), (52) pairs with (55), (51) pairs with (54), (49) pairs with (53), (51) pairs with (55), and (51) pairs with (53). Consequently, when considering these pairs, Equation (48) is confirmed to be oscillatory.

Example 3. Consider the special case of (48), which simplifies to

$$\left(s^2 \left(s^2 \left(x(s) + p_0 x \left(\frac{2}{3}s \right) \right)' (s) \right)' \right)' + q_0 s \left[x \left(\frac{1}{4}s \right) + x \left(\frac{1}{5}s \right) + x \left(\frac{1}{6}s \right) \right] = 0, \quad (56)$$

Clearly,

$$n = 3, r_1(s) = r_2(s) = s^2, \eta_{\max}(s) = \frac{1}{4}s, \eta_{\min}(s) = \frac{1}{6}s, \tau(s) = \frac{2}{3}s, q_i(s) = q_0 s.$$

Thus, we obtain

$$\pi_1(s) = \pi_2(s) = \frac{1}{s}, \pi_3(s) = \pi_*(s) = \frac{1}{2s^2}.$$

By defining $w(s) = z(s) / \pi_*(s)$, Equation (48) is transformed into its canonical form:

$$w'''(s) + \frac{2q_0}{s} \left[x \left(\frac{1}{4}s \right) + x \left(\frac{1}{5}s \right) + x \left(\frac{1}{6}s \right) \right] = 0.$$

Additionally, we have

$$g_i(s) = \frac{2q_0}{s}, \tilde{g}_i(s) = \frac{2q_0}{s}, G_1(s) = 77q_0 \frac{1}{s^3}, G_2(s) = \frac{37q_0}{60} \frac{1}{s^3}, G_3(s) = \frac{77q_0}{2} \frac{1}{s^3}.$$

Choosing $\zeta(s) = \frac{1}{3}s$, we observe that $\eta_{\max}(s) = \frac{1}{4}s < \frac{1}{3}s = \zeta(s)$ and $\tau^{-1}(\zeta(s)) = \frac{1}{2}s < s$, so Condition (20) holds if

$$q_0 > 3.4739.$$

By choosing $\delta(s) = \frac{1}{2}s$, we see that $\delta(s) < s$ and $\eta_{\max}(s) = \frac{1}{4}s < \frac{1}{3}s = \tau(\delta(s))$, so Condition (21) holds if

$$q_0 > 13.091.$$

Condition (23) is satisfied if

$$q_0 > 4.8089.$$

By setting $\eta_0 = 3$, we have $(\eta_i^{-1}(s))' = (\eta_i^{-1}s)' = \frac{1}{\eta_i} \geq \eta_0 > 0, i = 1, 2, 3$, so Condition (28) is satisfied when

$$q_0 > 0.18576.S$$

Condition (32) holds if

$$q_0 > 0.41558.$$

Similarly, Condition (40) is satisfied when

$$q_0 > 0.27278.$$

Furthermore, Condition (46) holds if

$$q_0 > 0.029172.$$

Therefore, if $q_0 > 13.091$, all conditions are satisfied, and all solutions of Theorem (7) are oscillatory when $q_0 > 13.091$.

7. Conclusions

This study presents a comprehensive investigation into the oscillatory and asymptotic behavior of neutral third-order differential equations by transforming them from noncanonical to canonical forms. This transformation significantly simplifies the analytical process, reducing the complexity of the problem from four cases to two. We have established specific conditions that effectively rule out the existence of Kneser-type solutions and positive solutions under certain constraints. Building on these findings, we developed new criteria that guarantee the oscillation of all solutions to the equations studied. This contribution is crucial for advancing the theoretical framework of neutral differential equations and provides a solid foundation for future analyses. Additionally, we provided illustrative examples that demonstrate the practical applicability and theoretical significance of our criteria. These examples highlight the effectiveness of our approach in addressing complex problems related to neutral differential equations. The results obtained in this study extend existing theoretical frameworks within the field, opening new avenues for further research. We propose that future studies explore the application of our methods to higher-order equations, particularly those of odd order $n \geq 3$, as well as to more general forms, such as

$$\left(r_2(s) \left(r_1(s) (z'(s))^\alpha \right)' \right)' + \sum_{i=1}^n q_i(s) x^\alpha(\eta_i(s)) = 0.$$

These extensions hold the potential to uncover deeper insights into the oscillatory and asymptotic properties of complex systems, thereby contributing to the ongoing advancement of the field.

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